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The Josephson effect --- theory and phenomena

Discussion of the Josephson effect in five parts:

1. Theory and phenomena
2. The RSJ model
3. Magnetic field effects in extended junctions
4. Fluctuations and quantum tunneling
5. Beyond tunnel junctions (SNS, microbridges, SFS, …)

Josephson Effect Physics Letters 1, 251 (1962)

POSSIBLE NEW EFFECTS IN SUPERCONDUCTIVE TUNNELLING *

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Received 8 June 1962

We here present an approach to the calculation of tunnelling currents between two metals that is sufficiently general to deal with the case when both metals are superconducting. In that case new effects are predicted, due to the possibility that electron pairs may tunnel through the barrier leaving the quasi-particle distribution unchanged.

Our procedure, following that of Cohen et al. [1], is to treat the term in the Hamiltonian which transfers electrons across the barrier as a perturbation. We assume that in the absence of the transfer term there exist quasi-particle operators of definite energies, whose corresponding number operators are constant.

A difficulty, due to the fact that we have a system containing two disjoint superconducting regions, arises if we try to describe quasi-particles by the usual Bogoliubov operators [2]. This is because states defined as eigenfunctions of the Bogoliubov quasi-particle number operators contain phase-coherent superpositions of states with the same total number of electrons but different numbers in the two regions. However, if the regions are independent these states must be capable of superposition with arbitrary phases. On switching-on the transfer term the particular phases chosen will affect the predicted tunnelling current. This behaviour is of fundamental importance to the argument that follows. The neglect, in the quasi-particle approximation, of the collective excitations of zero energy [3] results in an unphysical restriction in the free choice of phases, but may be avoided by working with the projected states with definite numbers of electrons on both sides of the barrier. Corresponding to these projections we use operators which alter the numbers of electrons on the two sides by definite numbers **. In particular, corresponding to the Bogoliubov operators α_k^+ we use quasi-particle creation operators α_{ek}^+, α_{hk}^+ which respectively add or remove an electron from the same side as the quasi-particle and leave the number on the other side unchanged, and pair creation operators S_k^+ † which add a pair of electrons on one side leaving the quasi-particle distribution unchanged. The Hermitean conjugate destruction operators have similar definitions. The S operators, referring to macroscopically occupied states, may be treated as time dependent c-numbers ††, and we normalise them to have unit amplitude. Relations expressing electron operators in terms of quasi-particle operators, equal-time anticommutation relations and number operator relations may be derived from those of the Bogoliubov theory by requiring both sides of the equations to have the same effect on N_l and N_r, the numbers of electrons on the two sides of the barrier. For example,

$$a_{k\uparrow}^+ = u_k\, \alpha_{ek0}^+ + v_k\, \alpha_{hk1}\,, \qquad \alpha_{ek}^+\alpha_{ek} = n_k\,, \qquad \alpha_{ek}^+\alpha_{hk} = S_k^+\, n_k\,. \tag{1}$$

Noting that the Bogoliubov Hamiltonian is $H - \lambda N$ (λ = chemical potential), we take our unperturbed Hamiltonian to be

$$H_0 = \sum_k n_k E_k + \lambda_l N_l + \lambda_r N_r\,,$$

where E_k is the quasi-particle energy in the Bogoliubov theory, and derive the commutation relations

$$[H_0, \alpha_{ek}^+] = (E_k + \lambda_k)\,\alpha_{ek}^+\,,$$
$$[H_0, \alpha_{hk}^+] = (E_k - \lambda_k)\,\alpha_{hk}^+\,, \tag{2}$$
$$[H_0, S_k^+] = 2\,\lambda_k\, S_k^+\,.$$

In the presence of tunnelling the Hamiltonian is $H_0 + H_T$, where H_T expressed in electron operators is

$$\sum_{l,r} (T_{lr}\, a_l^+ a_r + T_{rl}\, a_r^+ a_l)\,.$$

If we describe the time dependence of operators by the interaction picture [6], equations (2) imply that the α and S operators have exponential time dependence. The current operator in the Heisenberg picture is related to that in the interaction picture according to

$$J_H(t) = U^{-1}(t)\, J_{\text{int}}(t)\, U(t)\,,$$

where

$$U(t) = \lim_{\varepsilon \to 0+} \{T \exp\,(-i/\hbar \int_{-\infty}^{t} e^{\varepsilon t'} H_T(t')\, dt')\}\,.$$

Here H_T is expressed in the interaction picture and $U(t)$ can be evaluated by writing H_T in terms of quasi-particle operators and using the method of Goldstone [6]. We also express

$$J_{\text{int}}(t) = ie/\hbar\,[H_T, N_r] \quad ^{1)}$$

in terms of quasi-particle operators, and by retaining only those terms in $J_H(t)$ which can be expressed in accordance with (1) as products of S and number operators obtain an expression equivalent to the usual one, of the form

$$J_H = J_0 + \tfrac{1}{2}\, J_1\, S_l^+ S_r + \tfrac{1}{2}\, J_1^* S_r^+ S_l\,. \tag{3}$$

To second order in H_T, J_0 is similar to the expression of Cohen et al. [1], and reduces for the same reasons to the usual one obtained by neglecting coherence factors. The remaining terms oscillate with frequency $\nu = 2eV/h$ ($V = \lambda_l - \lambda_r$ being the applied voltage), owing to the time dependence of the S operators. J_1 is proportional to the effective matrix element for the transfer of electron pairs across the barrier without affecting the quasi-particle distribution, and typical terms are of the form

$$2\,i\,e\,u_l v_l u_r v_r T_{lr} T_{-l,-r} \,[(1 - n_{l0} - n_{r1})$$
$$\times \{P\frac{1}{eV - E_l - E_r} + \pi i\delta(eV - E_l - E_r)\}$$
$$- (n_{l0} - n_{r0})\,\{P\frac{1}{eV + E_l - E_r} + \pi i\delta(eV + E_l - E_r)\}] \tag{4}$$

where $-k$ denotes the state paired with k. The second and fourth terms result from processes involving real intermediate states, and can be regarded as fluctuations in the normal current due to coherence effects. We note that the first term remains finite at zero temperature and zero applied voltage. From (3) our theory predicts that

(i) At finite voltages the usual DC current occurs, but there is also an AC supercurrent of amplitude $|J_1|$ and frequency $2\,eV/h$ (1 μV corresponds to 483.6 Mc/s).

(ii) at zero voltage, J_0 is zero, but a DC supercurrent of up to a maximum of $|J_1|$ can occur.

Applied r.f. fields can be treated by noting that the oscillations in V frequency-modulate the supercurrent. Thus if a DC voltage V on which is superimposed an AC voltage of frequency ν is applied across the barrier, the current has Fourier components at frequencies $2eV/h \pm n\nu$, where n is an integer. If for some n, $2eV/h = n\nu$, the supercurrent has a DC component dependent on the magnitude and phase of the AC voltage. Hence the DC characteristic has a zero slope resistance part over a range of current dependent on the magnitude of the AC voltage.

Equivalent quantum-mechanical explanations of these effects can be given. For example (i) is due to the transfer of an electron pair across the barrier with photon emission, leaving the quasi-particle distribution unchanged. Consequently the photon frequency is not broadened by the finite quasi-particle lifetimes occurring in real superconductors. (ii) is due to pair transfer without photon emission. The linear dependence of the current on the matrix element is due to the fact that the process involves macroscopically occupied states between which phase relationships can occur.

The possibility of observing these effects depends on the value of $|J_1|$. At low temperatures and voltages the first term of (4) dominates, and in the presence of time-reversal symmetry all contributions to it are in phase. $|J_1|$ is then equal to the current flowing in the normal state at an applied voltage equal to π times the energy gap, assumed to be the same on both sides. At higher temperatures the third term reduces $|J_1|$, and at high frequencies the effects are reduced by the capacitance across the barrier. Magnetic fields, and currents in the films destroy the time-reversal symmetry and reduce $|J_1|$. The effects may be taken into account approximately by replacing (3) by

$$j = j_0 + \tfrac{1}{2}\, j_1\, \psi_l^* \psi_r + \tfrac{1}{2}\, j_1^* \psi_r^* \psi_l\,,$$

where j is the tunnelling current density, and ψ_l, ψ_r the effective superconducting wave functions [7] in the films on the two sides. This formula predicts that in very weak fields diamagnetic currents will screen the field from the space between the films, but with a large penetration depth owing to the smallness of j_1. In larger fields, owing to the existence of a critical current density, screening will not occur; the phases of the supercurrents will vary rapidly over the barrier, causing the maximum total supercurrent to drop off rapidly with increasing field. Anderson [8] has suggested that the absence of tunnelling supercurrents in most experiments hitherto performed may be due

* Work supported by Trinity College, Cambridge, and the Department of Scientific and Industrial Research.

** We shall use subscripts l and r to distinguish operators on the two sides, and k to denote an operator referring to either side.

† These are equivalent to the operators which change $|N\rangle$ to $|N+2\rangle$ in the theory of Gor'kov [4].

†† Cf. N. N. Bogoliubov et al. [5]. The phase of an S operator is related to the orientation of the plane containing the pseudospin operators [3]. Physical observables cannot depend on the phase of a single S operator, but they can depend on the relative phases of the S operators associated with two superconducting regions, as in the phenomena dealt with here.

to the earth's field acting in this way. Cancellation of supercurrents would start to occur when the amount of flux between the films, including that in the penetration regions, became of the order of a quantum of flux $hc/2e$. This would occur for typical films in a field of about 0.1 gauss. Such a field would not be appreciably excluded by the critical currents obtainable in specimens of all but the highest conductivity.

When two superconducting regions are separated by a thin normal region, effects similar to those considered here should occur and may be relevant to the theory of the intermediate state.

I am indebted to Dr. P. W. Anderson and Prof. A. B. Pippard for stimulating discussions.

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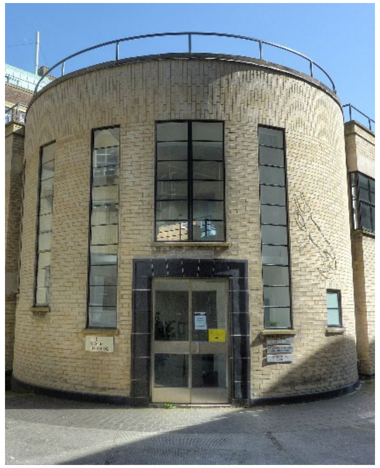
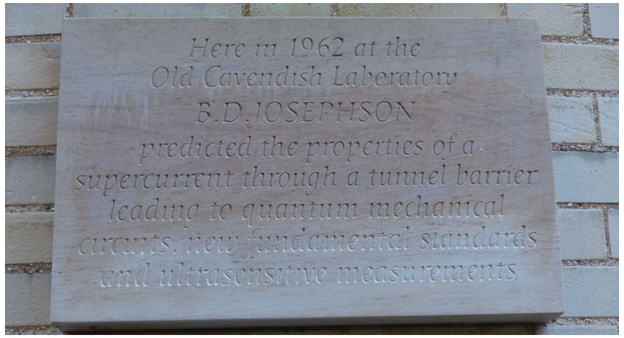




<u>Josephson Effect</u> (1962)   Physics Letters 1, 251 (1962)

Microscopic theory $\Longrightarrow$ not from physical agreement or experiment

$$H = H_L + H_R + H_T \qquad H_T = \sum_{\ell,r} T(c_r^+ c_\ell + c_\ell^+ c_r)$$

$c's \longrightarrow \gamma_e's,\ \gamma_h's$ Josephson operators

γ_e^+ creates one unit of electronic charge and one qp
γ_h^+ removes one unit of electronic charge and adds one qp

S^+ creates pair   x $(e^{-i\theta})\psi_{BCS}$
S destroys pair   x $(e^{i\theta})\ \psi_{BCS}$

$$\gamma_{ek0}{}^+ = u_k c_{k\uparrow}{}^+ - v_k\, S^+\, c_{-k\downarrow}$$
$$\gamma_{hk0}{}^+ = u_k\, S\, c_{k\uparrow}{}^+ - v_k c_{-k\downarrow}$$
$$\gamma_{ek1}{}^+ = u_k c^+ + v_k\, S^+\, c_{k\uparrow}$$
$$\gamma_{hk1}{}^+ = u_k\, S\, c_{-k\downarrow}{}^+ + v_k c_{k\uparrow}$$

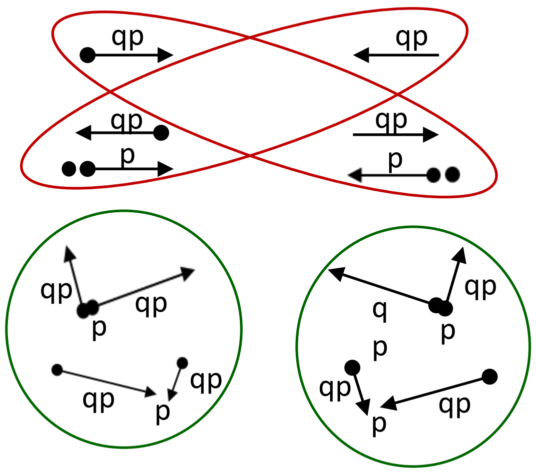
<u>Processes:</u>   <u>$L\rightarrow R$</u>   <u>$R\rightarrow L$</u>

(1) $\gamma_r^+\gamma_\ell$

(2) $(\gamma_r\gamma_\ell^+)(S_r^+S_\ell)$

(3) $(\gamma_r^+\gamma_\ell^+)S_\ell$

(4) $(\gamma_r\gamma_\ell)S_r^+$



Same qp states involved, pair transfer different

(1) $L \rightarrow R$ + (2) $R \rightarrow L$ $(S_r^+S_\ell)\sim e^{i\theta_r}e^{-i\theta_\ell} = e^{i\varphi}$

$$\varphi = (\theta_r - \theta_\ell)$$

(2) $L \rightarrow R$ + (1) $R \rightarrow L$ $(S_\ell^+S_r)\sim e^{i(\theta_\ell-\theta_r)} = e^{-i\varphi}$

(3) $L \rightarrow R$ + (3) $R \rightarrow L$ $(S_r^+S_\ell)\sim e^{i\theta_r}e^{-i\theta_\ell} = e^{i\varphi}$

(4) $L \rightarrow R$ + (4) $R \rightarrow L$ $(S_\ell^+S_r)\sim e^{i(\theta_\ell-\theta_r)} = e^{-i\varphi}$

<u>Cross terms</u> $e^{i\varphi}, e^{-i\varphi} \implies \sin\varphi\,, \cos\varphi$

$$I(V,T,t) = \sigma_o(V,T)V + I_1(V,T)\sin\varphi(t) + \sigma_1(V,T)\cos\varphi(t)V$$

(1) $\sigma_o V = G_{SS}V =$ qp tunneling (as before)

(2) $I_1 \sin\varphi =$ supercurrent = pair tunneling (current at V=0)

(3) $\sigma_1 V \cos\varphi =$ quasiparticle-pair interference term
(supercurrent but dissipative)

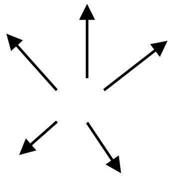
(2) and (3): Josephson effects

How did Josephson figure this out? Listening to lectures by P. Anderson (lectures on phase transitions)

2nd order phase transition – <u>broken symmetry</u>
lower symmetry than the Hamiltonian

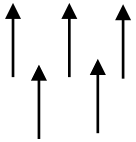
Ferromagnet:

$T > \theta$



Moments random

$T < \theta$



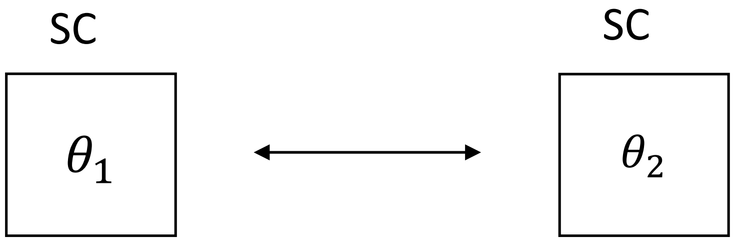
Moments aligned

"rotational invariance"

Superconductor: phase (Cooper pairs)

Isolated system: direction (FM) or phase (SC) arbitrary

Coupled systems: domain walls (FM) or interfaces (SC) force a dependence on the phase alignment

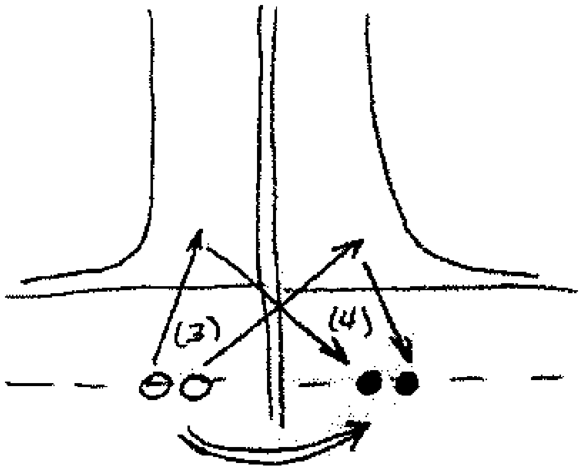


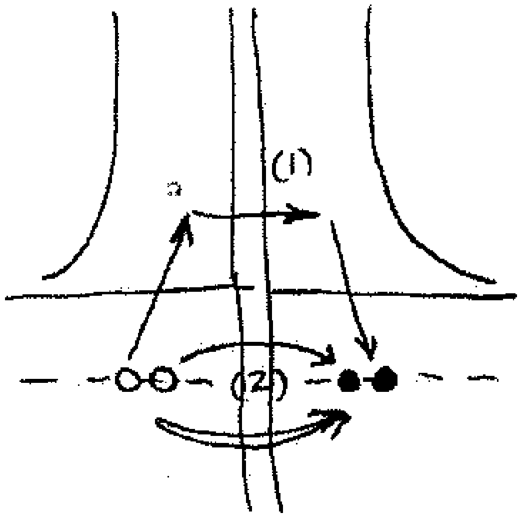
Junction properties should depend on $(\theta_1 - \theta_2)$

But then N must be indeterminate $(\Delta N\, \Delta\theta \gtrsim 1)$

(BCS)

This suggests that pairs could change sides with energy cost (no dissipation)





This was not believed by many, e.g. John Bardeen

Like multi-particle tunneling, it was thought that the rate should go as $(|T|^2)^2$ and be very low probability

Paper that introduces the Josephson operators

locity of sound in a transverse magnetic field.

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[6] In the numerical estimates that follow, we assume the Fermi energy in the absence of a magnetic field $\zeta_0 = \frac{1}{2}mv_0^2 = 5$ eV and $s_0 = 5\times10^5$ cm/sec.
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TUNNELING INTO SUPERCONDUCTORS*

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(Received June 18, 1962)

In a recent Letter, Cohen, Falicov, and Phillips[1] have discussed tunneling of electrons through a thin insulating layer between a normal and a superconducting metal on the basis of an effective Hamiltonian,

$$H = H_n + H_S + H_T, \tag{1}$$

where H_n and H_S are exact Hamiltonians for the normal and superconducting metals, respectively, and H_T is an operator which transfers electrons from one to the other:

$$H_T = \sum_{k,q,\sigma} (T_{kq} c_{k\sigma}^* c_{q\sigma} + T_{kq}^* c_{q\sigma}^* c_{k\sigma}). \tag{2}$$

Here k is a quantum number describing states in the normal metal, q refers to states in the superconductor, σ is the spin, and the c's are creation and destruction operators for normal quasi-particle states in both metals. By making use of the equations of motion, they derived an expression for the time rate of change of number of electrons in the superconductor $\langle N_S \rangle$ and thus the tunneling current. They find that the ratio of tunneling currents in superconducting and normal states depends only on the density of states in energy in the superconductor, as indicated by the experiments.[2]

We would like to discuss their derivation from a somewhat different point of view, which we feel brings out a little more clearly the connection with the semiconductor model of a superconductor and also with an earlier discussion of tunneling by the present author.[3] In the semiconductor model, one assumes that there is a set of normally occupied quasi-particle states below the gap and a set of normally unoccupied states above the gap, in one-to-one correspondence with those of the normal metal. At $T = 0°K$, states above are all unoccupied, those below occupied, but at a finite temperature electrons may be thermally excited to states above the gap, leaving holes in the normally occupied band. Electrons may be transferred from the normal to the superconducting metal into unoccupied states above the gap or into holes below the gap. Correspondingly, transfer in the reverse direction occurs from occupied states above the gap or from one of the filled states below the gap, leaving holes behind. It is the occupied states above the gap and the holes below which correspond to quasi-particle excitations of the superconductor.

What the author showed in his earlier Letter is that if there is a one-to-one correspondence between the quasi-particle excitations in normal and superconducting states, the only significant factor in the tunneling current is given by the density of states in energy. However, justification for the one-to-one correspondence and the definition of the quasi-particle states from microscopic theory was not given.

The quasi-particle states in a superconductor are usually defined by the Bogoliubov-Valatin transformation,[4]

$$\gamma_{q\uparrow}^* = u_q c_{q\uparrow}^* - v_q c_{-q\downarrow}, \tag{3a}$$

$$\gamma_{-q\downarrow}^* = u_q c_{-q\downarrow}^* + v_q c_{q\uparrow}, \tag{3b}$$

where $u_q^2 = 1 - v_q^2 = \frac{1}{2}(1 + \epsilon_q/E_q)$; $E_q = (\epsilon_q^2 + \Delta^2)^{1/2}$, and $-q$ indicates the time-reversal conjugate of q. These operators do not conserve particle number and are designed to operate on wave

Note added in proof: In a recent note, Josephson[7] uses a somewhat similar formulation to discuss the possibility of superfluid flow across the tunneling region, in which no quasi-particles are created. However, as pointed out by the author (reference 3), pairing does not extend into the barrier. so that there can be no such superfluid flow.

How to understand why this is not true? This is an approach by Waldram to try to make sense of this.

Time-dependent perturbation theory:

$$\psi = \sum_i a_i \psi_i \Rightarrow a_i = \langle \psi_i | \psi \rangle$$

$$\frac{d}{dt}|a_i|^2 = -\frac{i}{\hbar}\sum_j \left(a_j^* T_{ji} a_i - a_i^* T_{ij} a_j\right) = -\frac{i}{\hbar} T \sum_j \left(a_j^* a_i - a_i^* a_j\right)$$

*Usually, the system is near an eigenstate where some $a_j = 1$ (most probable state)
Then, $a_i \sim T$ (perturbation theory) and rate of transition to depend on density of available states $\Longrightarrow$ Golden Rule

$$\frac{d}{dt}|a_i|^2 = \frac{2\pi}{\hbar}|T|^2 N(E_i)$$ higher-order transitions

*But, if we consider all $a_i \approx$ same so just consider two states i and j $|a_i|^2 \approx |a_j|^2 \approx \frac{1}{2} \Longrightarrow a_i = a_j e^{i\theta}$

differ by phase factor only

$$\frac{d}{dt}|a_i|^2 \sim T|a_i|^2\left(e^{i\theta} - e^{-i\theta}\right)$$

$$\frac{d}{dt}|a_i|^2 \sim T \sin\theta$$

depends on just T as in quasiparticle tunneling

The key once again is the $\Delta N - \Delta\theta$ uncertainty relation stemming from the dynamic Cooper pairing in the ground state

Key points: rates scales as 1st power of matrix elements

correlated 2-step process $\sim|T|^2$ so allowable

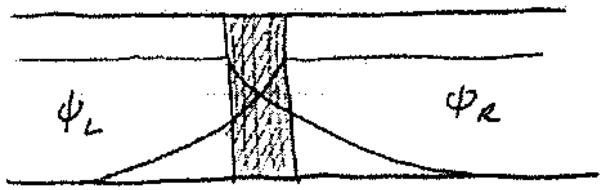
$\sin\theta$ comes out naturally

*Josephson current – 2nd order pair tunneling between states mixed in pair occupancy

KEY – <u>fixed phase</u> uncertain $\Longrightarrow$ Josephson tunneling

Pairs don't know or care which side they are on

Annother approach: Feynman treatment – two-level system model due to Rogovin & Scully (1974)



$$\psi_L = \sqrt{n_L}\, e^{i\theta_L} \qquad \psi_R = \sqrt{n_R}\, e^{i\theta_R}$$

Uncoupled superconductors:

$$i\hbar \frac{\partial \psi_L}{\partial t} = E_L \psi_L \qquad i\hbar \frac{\partial \psi_R}{\partial t} = E_R \psi_R \qquad E_L - E_R = 2eV$$

Add tunneling: $H = H_L + H_R + H_T$

$$i\hbar \frac{\partial \psi_L}{\partial t} = E_L \psi_L + K\psi_R$$

where $K = \langle \psi | H_T | \psi \rangle$ is the tunneling rate

$$i\hbar \frac{\partial \psi_R}{\partial t} = E_R \psi_R + K\psi_L$$

4 coupled differential equations $\dot{n}_L, \dot{n}_R, \dot{\theta}_L, \dot{\theta}_R$

$$\dot{n}_L = -\dot{n}_R = \frac{2}{\hbar} K \sqrt{n_L n_R} \sin\theta$$

$$\dot{\theta} = \dot{\theta}_L - \dot{\theta}_R = \frac{2eV}{\hbar}$$

$$\Delta E \sim 2eV$$

Josephson supercurrent: $I = I_c \sin\theta$

Josephson relation: $V = \frac{\hbar}{2e} \dot{\theta}$

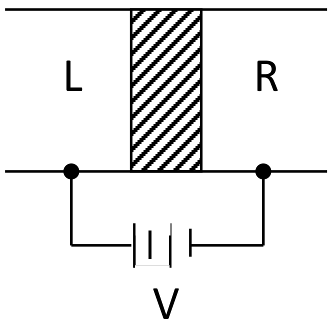
$$\psi \sim e^{-i(\Delta E/\hbar)t} = e^{-i\omega t}$$

This is Specific to tunnel junctions

Seen in all kinds of systems
SIS, $S_s S$
SNS
μ bridges
proximity bridges
point contact

(weak-coupled superconductors)

Josephson Tunneling



$$H = H_L + H_R + H_T$$

$$I(V,T,t) = \sigma_o(V,T)V + I_1(V,T)\sin\varphi(t) + \sigma_1(V,T)V\cos\varphi(t)$$

where $\varphi = \theta_L - \theta_R - \dfrac{2e}{\hbar}\int \vec{A}\cdot\overrightarrow{d\ell}$

$$I_{qp} = \sigma_o V = \frac{4\pi e}{\hbar}|T|^2 \int dE \; N_L(E)\, N_R(E+eV)\, [F_L(E) - F_R(E+eV)]$$

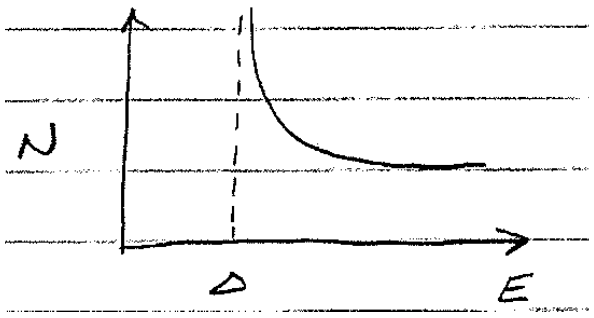
$$I_1 = \frac{4e}{\hbar}|T|^2\, P \iint dE\, dE' \; \mathrm{P_L(E)} P_R(E')\, \frac{F_L(E) - F_R(E')}{E - E' + eV}$$

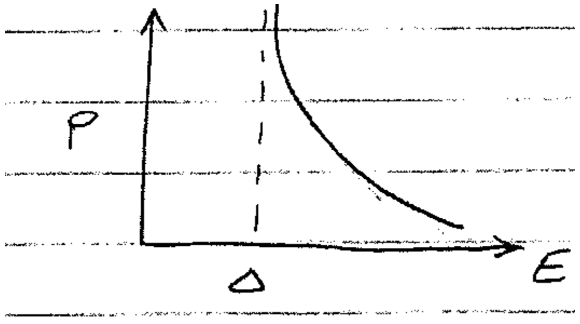
$$N(E) = \frac{E}{|\xi_E|} = \frac{E}{\sqrt{E^2 - \Delta^2}}$$

"qp density of states"

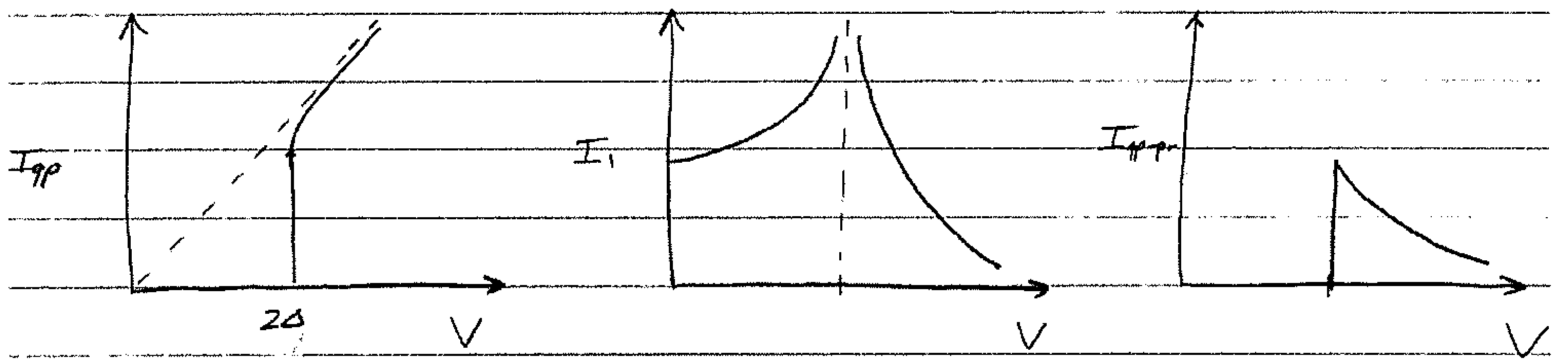
$$P(E) \equiv \frac{\Delta}{|\xi_E|} = \frac{\Delta}{\sqrt{E^2 - \Delta^2}}$$

"pair density of states"





$$I_{qp-pv} = \sigma_1 V = \frac{4\pi e}{\hbar}|T|^2 P \iint dE\, dE'\, P_L(E) P_R(E+eV)[F_L(E) - F_R(E+eV)]$$

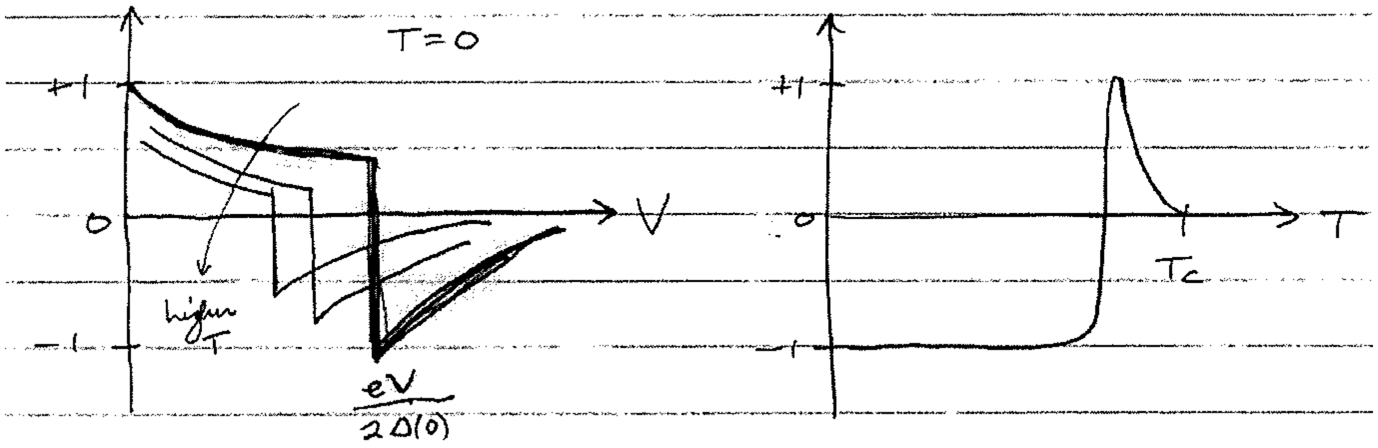


Cosϕ term

BCS theory: $\varepsilon = \frac{\sigma_1}{\sigma_o} = +1$ as $V \to 0,\ T \to 0$

Experiment: $-1 \le \varepsilon \le 1$ function of V & T

ε



Not really understood – sensitive to model, tunneling details

Maybe not very important but comes up periodically to try to explain something

Josephson theory (1) Explicitly get sinϕ, Josephson relation

(2) Not specific to tunnel junctions ⟹ weak coupling SC

(3) Cannot get cosϕ or T, V dependence of I (ignores qp's)

Extensions:

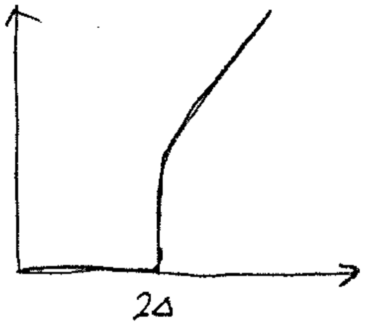
1. Werthamer's formula (frequency dependent voltages)

$$I(t) = I_m \left[e^{-\frac{1}{2}i\phi(t)} \int_{-\infty}^{t} dt'\, e^{1/2\,\phi(t')}\, j_1(t-t') + e^{\frac{1}{2}i\phi(t)} \int_{-\infty}^{t} dt'\; e^{1/2\,\phi(t)'}\, j_2(t-t') \right]$$

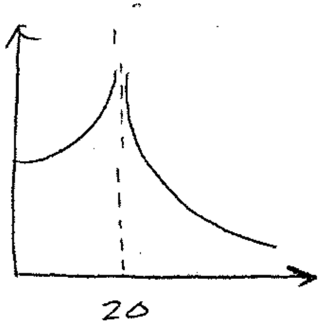
normal current supercurrent

$H_T \rightarrow \delta\psi \rightarrow I$ $\quad I(V) = I_m\tilde{J}_1 + (-R_e\tilde{J}_2(\omega))\sin\varphi + (I_m\tilde{J}_2)\cos\varphi$

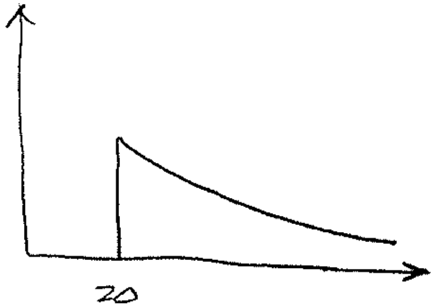
$\sigma_o V$



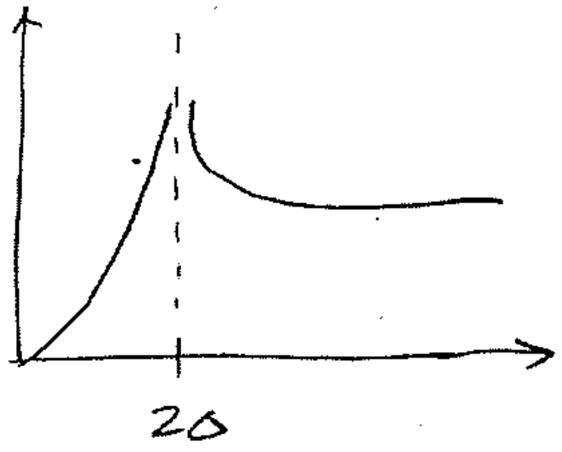
I_1



$\sigma_1(V)V$



$R_e\tilde{J}_1$ does not appear in Josephson original calculation



Most important in dealing with high frequency effects – biased beyond gap (parametric amplifiers)

Strong Josephson effects beyond the gap

Only works for insulator tunnel junctions

2. Josephson thermal Green's function treatment

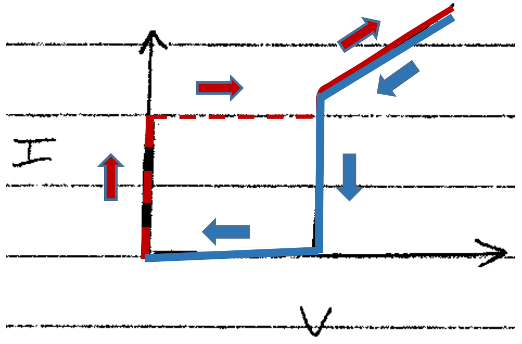
1st principles calculation

Can be generalized to any weak link

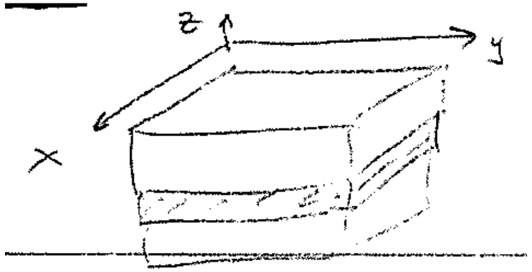
Supercurrent

$$I = I_c \sin \phi$$

DC: $V = \frac{\hbar}{2e}\dot{\phi} = 0$ $\quad \phi = \text{constant}$



Local: $J(x, y) = J_c(x, y) \sin \phi\,(x, y)$

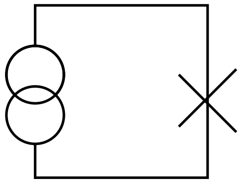


Key is periodicity – not exact form

Tunnel oxide barriers JJ $\Rightarrow$ very accurately sinusoidal but other junctions are not

Current-phase relation: $\quad I(\phi) = f(\phi)$ periodic

Phase coherence extends across barrier
Current depends on phase, which is set by the external circuit

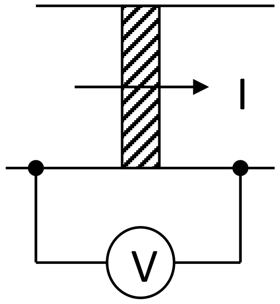


Phase adjusts according to applied current $I < I_c$

$$\phi = \sin^{-1}\left(\frac{I}{I_c}\right)$$

<u>Josephson coupling energy</u> :

Phase-coupling lowers energy

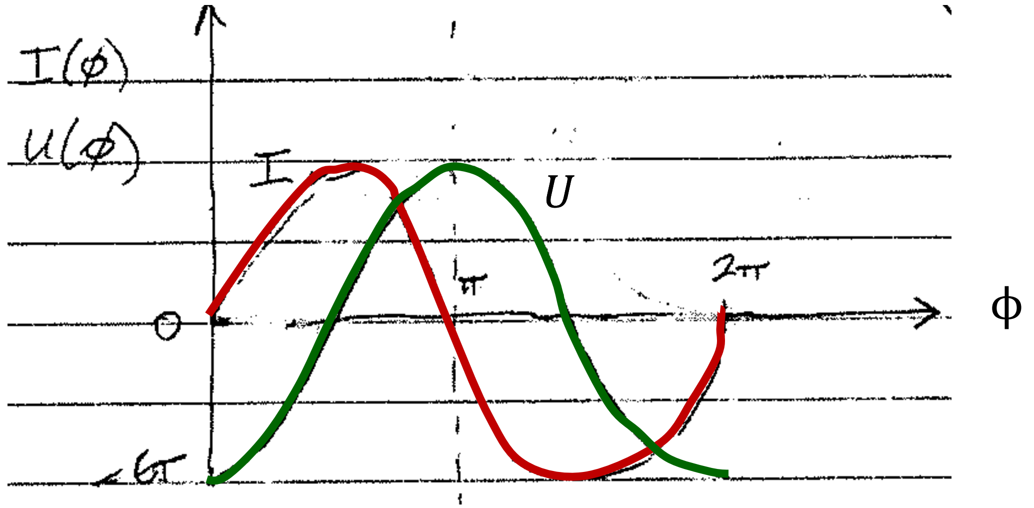


$$dU = IV dt$$

$$= \frac{I\hbar}{2e} \dot{\phi} dt$$

$$d\phi$$

$$\therefore \quad \frac{dU}{d\phi} = \frac{I\hbar}{2e}$$



Bias current: $U(\phi) = \frac{\hbar}{2e} I\phi$ I constant

Supercurrent: $dU = \frac{\hbar I_C}{2e} \sin\phi \, d\phi$ I periodic in phase

$$U(\phi) = -\frac{\hbar I_c}{2e} \cos\phi$$ assuming $U\left(\frac{\pi}{2}\right) = 0$

$$= -E_J \cos\phi$$

↑ no supercurrent

$\underline{Ic}$	$\underline{E_J}$	$T_{eff} =$ $\underline{E_J/k_B}$
1mA	2eV	(23,000)K
1µV	2meV	23K
1nV	2µeV	23mK

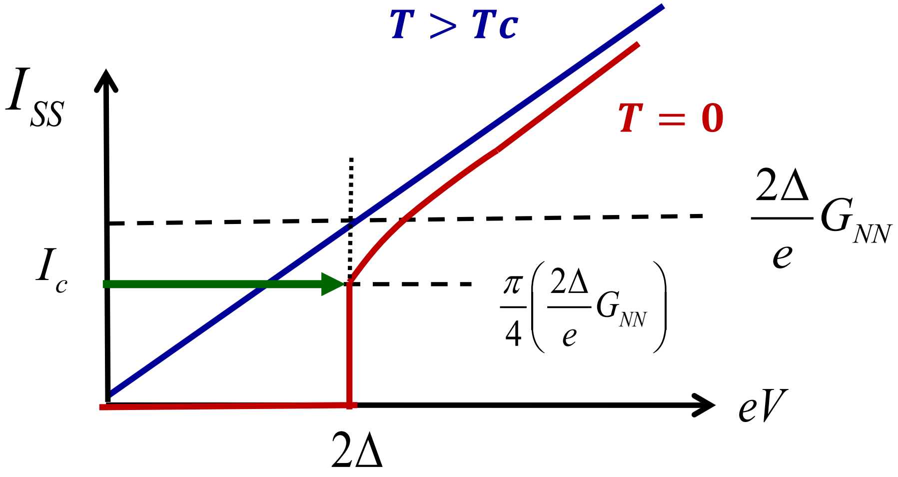
Josephson coupling energy $E_J = \left(\frac{\hbar I_1}{2e}\right) = \left(\frac{I_1 \Phi_o}{2\pi}\right)$

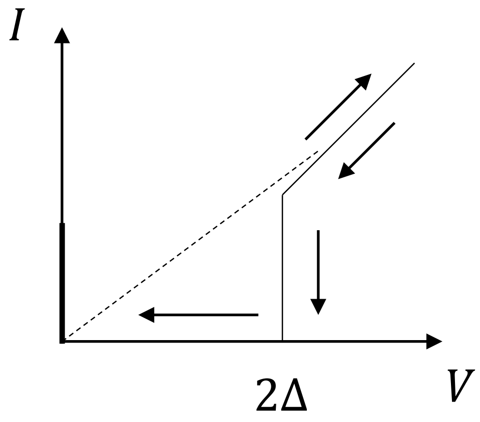
Variation of Ic with V, T

Special cases : $\Delta_1 = \Delta_2 = \Delta, \quad V = 0$

$$I_c = \frac{\pi\Delta}{2eR_n}$$

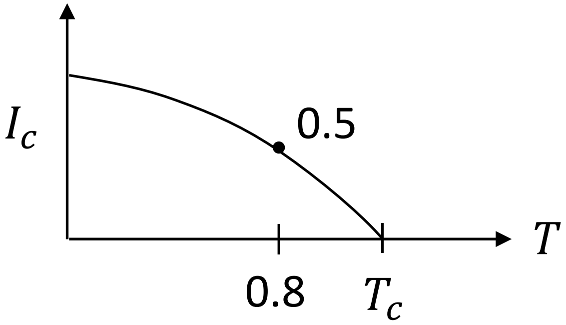
same as jump in qp tunneling

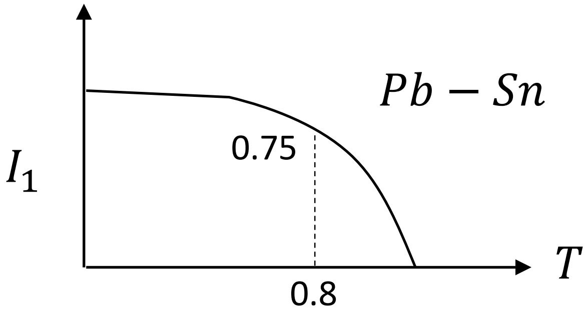




$\Delta_L \neq \Delta_R$

$$I_c(0) = \frac{I}{eR_N}\left(\frac{\Delta_L \Delta_R}{\Delta_L + \Delta_R}\right) K \left|\frac{\Delta_L - \Delta_R}{\Delta_L + \Delta_R}\right|$$





$\Delta_{Pb} = 2\Delta_{Sn}$

- Strong – coupling: reduces Ic

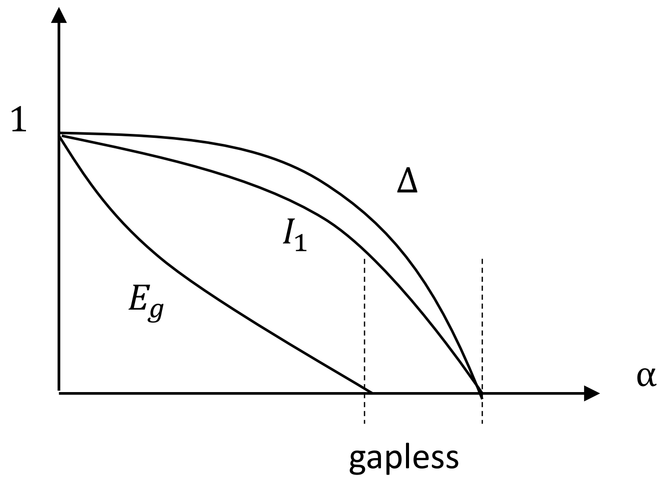
Nb	75%
Pb	78%
Sn	91%
Al	100%

- Magnetic impurities: reduces Ic

  (similar to effect on N, affects P)

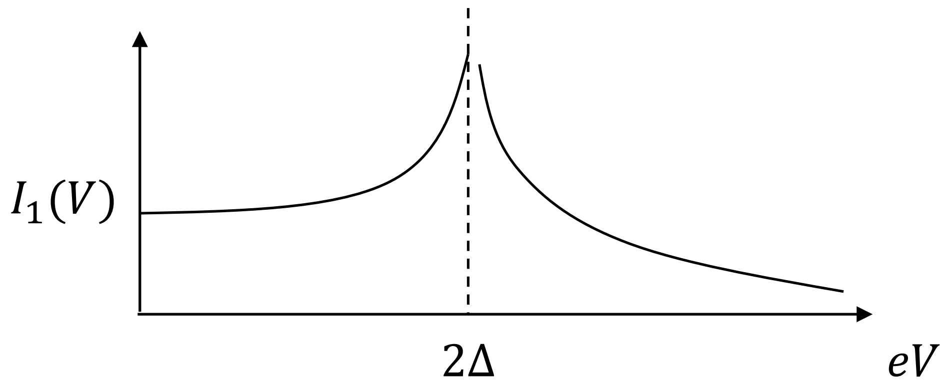
  Pairbreaking by spin-flip scattering

  In film, drops off roughly as Δ (order parameter)

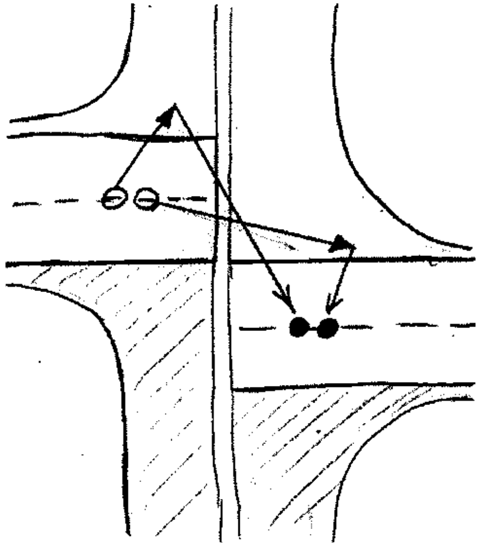


In barrier, drop is dramatic!

- Voltage (frequency)



Riedel singularity – "resonance between qp and pair tunneling", intermediate states highly probable



Important for operation of devices near or above 2Δ.

N.B. Josephson effects <u>can</u> occur for $\hbar\omega_J > 2\Delta$